Q.P. 67562

MATHEMATICS SOLUTION

(MAY 2019 SEM 4 MECHANICAL)

Q1. (a) Write the dual of the given LLP **(5M)**

Maximise $Z=4x_1+9x_2+2x_3$

Subjected to: $2x_1+3x_2+2x_3 \le 7,$

$3x_1-2x_2+4x_3=5,$

$x_1,x_2,x_3 \ge 0$

Solution:

Since the problem is of maximisation type, the constraints must be expressed in less than or equal to form.

$2x_1+3x_2+2x_3 \le 7$

$3x_1-2x_2+4x_3=5$

i.e. $3x_1-2x_2+4x_3 \ge 5$ and $3x_1-2x_2+4x_3 \le 5$

i.e. $-3x_1+2x_2-4x_3 \le -5$ and $3x_1-2x_2+4x_3 \le 5$

hence the given problem becomes,

Maximise $Z=4x_1+9x_2+2x_3$

Subjected to: $2x_1+3x_2+2x_3 \le 7,$

$-3x_1+2x_2-4x_3 \le -5$

$3x_1-2x_2+4x_3 \le 5,$

$x_1,\ x_2,\ x_3 \ge 0$

Since the last constraint in the primal is an equality y_3 must be unrestricted. Let y_1, y_2',y_2'' be the associated non negative variables of the dual. Then the dual is

$w = 7y_1+ 5y_2'- 5y_2''$

$2y_1+ 3y_2'- 3y_2'' \ge 4$

$3y_1+ 2y_2'- 2y_2'' \ge 9$

$2y_1+ 4y_2'- 4y_2'' \ge 2$

Putting $y_2'- y_2'' = y_2$, where y_2 is unrestricted, the required dual is

$w = 7y_1+ 5y_2$

$2y_1+ 3y_2 \ge 4$

$3y_1+ 2y_2 \ge 9$

$2y_1+ 4y_2 \ge 2$

$y_1 \ge 0$; y_2 is unrestricted.

(b) If X is a random variable with probability density function **(5M)**

f(x)= $\begin{cases} \boldsymbol{xk; 0 \le x \le 2} \\ \boldsymbol{2k; 2 \le x \le 4} \\ \boldsymbol{6k; 4 \le x \le 6} \end{cases}$

find k, expectation and P$(\boldsymbol{1 \le x \le 3})$.

Solution:

Since, $\int_{-\infty}^{\infty} f(x).dx = 1$

$\int_0^2 xk.dx + \int_2^4 2k.dx + \int_4^6 6k.dx = 1$

$\left(\frac{x^2}{2}k\middle|x = 0\ to\ 2\right) + (2kx|x = 2\ to\ 4) + (6xk|x = 4\ to\ 6) = 1$

(4/2) k + (8k – 4k) + (36k – 24k) = 1

2k + 4k + 12k = 1

k = 1/18

f(x)= $\begin{cases} \frac{x}{18}; 0 \le x \le 2 \\ \frac{1}{9}\ ; 2 \le x \le 4 \\ \frac{1}{3}\ ; 4 \le x \le 6 \end{cases}$

P$(1 \le x \le 3)$ = $\int_1^3 (\frac{x}{18} + \frac{1}{9}).dx$

= $\left(\frac{x^2}{2*18} + \frac{x}{9}\middle|x = 1\ to\ 3\right)$

= (3*3)/(18*2) + (3/9) – (1*1)/(18*2)-(1/9)

= 4/9

E(X) = $\int_{-\infty}^{\infty} x.f(x).dx$

= $\int_0^2 \frac{x^3}{18}.dx + \int_2^4 \frac{x}{9}.dx + \int_4^6 \frac{x}{3}.dx$

= $\left(\frac{x^4}{4*18}\middle|x = 0\ to\ 2\right) + \left(\frac{x^2}{2*9}\middle|x = 2\ to\ 4\right) + \left(\frac{x^2}{2*3}\middle|x = 4\ to\ 6\right)$

= $\frac{2^4}{4*18} + \frac{4^2}{2*9} - \frac{2^2}{2*9} + \frac{6^2}{2*3} - \frac{4^2}{2*3}$

= $\frac{38}{9}$

(c)A tyre company claims that the life of the tyres have mean 42,000 kms with standard deviation of 4,000 kms. A change in the production process is believed to a result in better product. A test sample of 81 new tyres has a mean life 42,500 kms. Test at 5% level of significance that the new product is significantly better than old one. (5M)

Solution:

H_o: μ= 42000 km.
H_1: μ > 42000 km.
Here n = 81,

Test statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{42500-42000}{4000/\sqrt{81}} = 1.125$
Alternate hypothesis shows, this is right tailed test.
α= 0.05, Z_α = 1.64 which is the critical value.

Decision :

Since $Z_{ca1} < Z_\alpha$ => Zeal lies in the acceptance region

Hence H_0 is accepted and H_1 is rejected

New product is not significantly better than the current one.

(d) Find the minimal polynomial of A=$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Is A derogatory. (5M)

Solution:

$|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$(2 - \lambda)*[(3 - \lambda)*(2 - \lambda) - 2] - 2\,[2 - \lambda - 1] + 1\,[2 - 3 + \lambda] = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\lambda = 1, 1, 5$

Let us now find minimal polynomial of A. We know that each characteristic root of A is a root of the minimal polynomial of A. So if f(x) is the minimal polynomial of A, then (x-1) and (x-5) are the factors of f(x)

Let us see whether $(x-1)(x-5) = x^2 - 6x + 5$ annihilates A

$$A^2 - 6A + 6I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - 6\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - \begin{bmatrix} 12 & 12 & 6 \\ 6 & 18 & 6 \\ 6 & 12 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, f(x) is monic polynomial of lowest degree that annihilates A. Hence f(x) is minimal polynomial of A. Since its degree is less than the order of A, A is derogatory.

Q2. (a) Use Big M method to solve the following LLP **(6M)**

Minimise $Z=2x_1+x_2$

Subjected to: $3x_1+x_2 = 3,$

$4x_1+3x_2 \geq 6,$

$x_1+2x_2 \leq 3,$

$x_1,\ x_2 \geq 0$

Solution:

We have,

Maximise $z' = -z = -2x_1-x_2-0s_1-0s_2-MA_1-MA_2$ …………..(i)

Subject to $3x_1+x_2+0s_1+0s_2+A_1+0A_2 = 3$ …………………(ii)

$4x_1+3x_2-s_1+0s_2+0A_1+A_2 = 6$……………………(iii)

$x_1+2x_2+0s_1+s_2+0A_1+0A_2 = 3$……………………(iv)

Multiply (ii) and (iii) by M and (i)

Maximise $z' = (-2+7M)\ x_1 + (-1+4M)\ x_2 - Ms_1+0s_2-A_1-0A_2-9M$

$z' + (2-7M)\ x_1 + (1-4M)\ x_2 + Ms_1+0s_2+0A_1+0A_2 = -9M$

Iteration No.	Basic Var.	Coefficient of						R.H.S. Soln	Ratio
		x_1	x_2	s_1	s_2	A_1	A_2		
0	z'	2-7M	1-4M	M		0	0	0	
A_1 leaves	A_1	3*	1	0	0	1	0	3	1
x_1 enters	A_2	4	3	-1	0	0	1	6	1.5
	s_2	1	2	0	1	0	0	3	3
1	z'	0	1/3-5M/3	M	0		0	-2-2M	
A_2 leaves	x_1	1	1/3	0	0		0	1	3
x_2 enters	A_2	0	5/3*	-1	0		1	2	6/5
	s_2	0	5/3	0	1		0	2	6/5
2	z'	0	0	1/5	0			12/5	
	x_1	1	0	1/5	0			3/5	
	x_2	0	1	-3/5	0			6/5	
	s_2	0	0	1	1			0	

$x_1=3/5$ $x_2=6/5$ $z'_{max}= -12/5$ $z_{min}=12/5$

(b) Find e^A and 4^A. If A= $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ **(6M)**

Solution:

The characteristic equation of A is

$\begin{vmatrix} \frac{3}{2}-\lambda & 1/2 \\ 1/2 & \frac{3}{2}-\lambda \end{vmatrix}=0$

$(3/2-\lambda)^2 - ¼ = 0$

$9/4 - 3\lambda + \lambda^2 - ¼ = 0$

$\lambda^2 - 3\lambda + 2=0$

$(\lambda-1)*(\lambda-2)=0$

$\lambda= 1, 2$

1. For $\lambda=1$, $[A - \lambda I]X = 0$ gives

   $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $2R_2$ and $2R_1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By R_2-R_1 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1+x_2 = 0$

putting x_2=-t , we get x_1=t

$X_1= \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence the eigen values are 1, -1.

2. For $\lambda=2$, $[A - \lambda I]X = 0$ gives

   $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $2R_2$ and $2R_1$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By R_2-R_1 $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1+x_2 = 0$

$x_1=x_2$

putting x_2=t, we get x_1=t

$X_2= \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence the eigen values are 1, 1.

M = $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\qquad |M| = 2$

M^{-1} = $\frac{adj\ A^{-1}}{|M|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now, D=$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If f(A)= e^{A}, $\qquad$ f(D)=e^{D}=$\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$

If f(A)= 4^{A}, $\qquad$ f(D)=4^{D}=$\begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

e^{A} = M f(D) M^{-1}

$=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$

Similarly, replacing e by 4, we get

4^{A}=$\frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix}=\begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

(c) Verify Green's theorem for $\int_c\ (3x^2 - 8y^2)dx + (4y -\ 6xy)dy$ where C is the closed curve given by y □x^2; $y = \sqrt{x}$ **(8M)**

Solution:

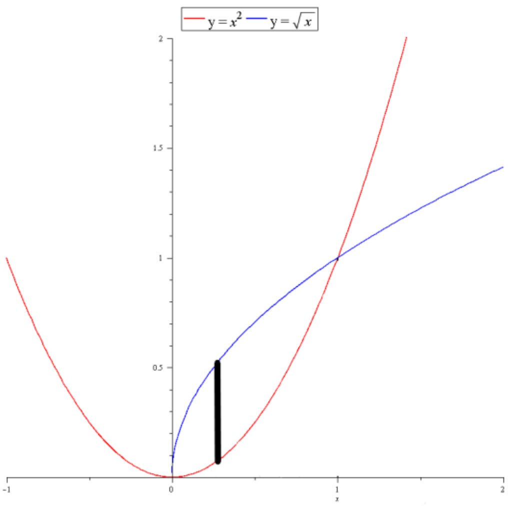
By Green's Theorem

$$\int_c P.dx + Q.dy = \iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\,.dx.dy$$

Here, P = $(3x^2 - 8y^2)$; Q= $(4y -\ 6xy)$

$\frac{\delta Q}{\delta x} = -6y$; $\frac{\delta P}{\delta y} = -16y$

Along, y=x^2 and dy=2x.dx and x varies from (0,1)



$$\int_c P.dx + Q.dy$$

$$= \int_0^1 (3x^2 - 8y^2).dx + (4y -\ 6xy).\mathrm{dy}$$

$$= \int_0^1 (3x^2 - 8x^4).dx + 2x(4x^2 - 6x^3).dx$$

$$= \left(\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\middle| x = 0\ to\ 1\right)$$

$$= 1 - \frac{8}{5} + \frac{8}{4} - \frac{12}{5} = -1$$

Along $y = \sqrt{x}$, $dy = \frac{1}{2\sqrt{x}}dx$

$$\int_c P.dx + Q.dy = \int_0^1 (3x^2 - 8x).dx + \left(4\sqrt{x} -\ 6x\sqrt{x}\right).\frac{1}{2\sqrt{x}}.dx$$

$$= \int_0^1 (3x^2 - 8x).dx + (2 - 3x).dx$$
$$= \left(\frac{3x^3}{3} - \frac{8x^2}{2} + 2x - \frac{3x^2}{2}\middle| x = 0\ to\ 1\right)$$
$$= 1 - 4 + 2 - \frac{3}{2} = \frac{-5}{2}$$

$$\iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}.dx.dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (-22y).dx.dy$$
$$= \int_0^1 \left(-11y^2\middle| y = x^2\ to\ \sqrt{x}\right).dx$$

$$= \int_0^1 -11x - (-11x^4).dx$$
$$= \int_0^1 -11x + 11x^4.dx$$
$$= \left(\frac{-11x^2}{2} + \frac{11x^5}{5}\middle| x = 0\ to\ 1\right)$$
$$= \frac{-33}{10}$$

Q3 (a) Prove that $\bar{F} = 2xyz^2 i + (x^2z^2 + z\cos yz)j + (2x^2yz + y\cos yz)k$ is a conservative field. Find ϕ such that $\bar{F} = \nabla.\phi$. Hence find the work done in moving an object in this field from (0,0,1) to (1,π/4,2). (6M)

Solution:

Curl $(\bar{F}) = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$

$= (2x^2z + \cos yz - yz\sin yz - 2x^2z + yz\sin yz - \cos yz)i + (4xyz - 4xyz)j + (2xz^2 - 2zx^2)k$
$= 0$

$\bar{F}$ is irrotatonal.

Since $\bar{F}$ is irrotatonal there exists a scalar function ϕ, such that $\bar{F} = \nabla.\phi$

$$2xyz^2 i + (x^2z^2 + z\cos yz)j + (2x^2yz + y\cos yz)k = \frac{\delta\phi}{\delta x} + \frac{\delta\phi}{\delta y} + \frac{\delta\phi}{\delta z}$$

$\frac{\delta\phi}{\delta x} = 2xyz^2$; $\frac{\delta\phi}{\delta y} = (x^2z^2 + z\cos yz)$; $\frac{\delta\phi}{\delta z} = (2x^2yz + y\cos yz)$

dϕ $= \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$

$= 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$
$= (2xyz^2dx + x^2z^2dy + 2x^2yzdz) + (z\cos yz\,dy + y\cos yz\,dz)$
$= d(x^2yz^2 + \sin yz)$

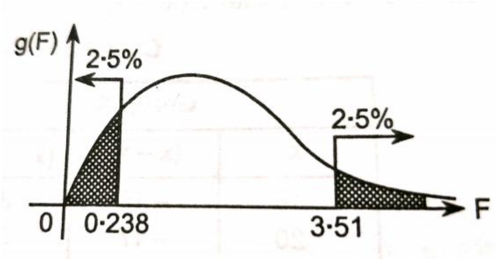
Φ=$x^2yz^2 + \sin yz$

Now, Work done = $\int_c \bar{F}.d\bar{r} = \int_c d(x^2yz^2 + \sin yz$

$= \left(x^2yz^2 + \sin yz\middle|(0,0,1) to \left(1,\frac{\pi}{4},2\right)\right)$
= π +1

(b)The standard deviations calculated from two random samples of sizes 9 and 13 are 1.99 and 1.9. Can the samples be regard as drawn from the normal populations with same standard Deviations.
(Given: $F_{0.025}$ = 3.51 with d.o.f. 8 & 12 and $F_{0.025}$ = 4.20 with d.o.f. 12 & 8.) **(6M)**
Solution:
Null Hypothesis Ho: $\sigma_1^2 = \sigma_1^2$
Alternative Hypothesis Ha: $\sigma_1^2 \neq \sigma_1^2$

Calculations of Test Statistic: $F=\frac{n_1 s_1{}^2/(n_1-1)}{n_2 s_2{}^2/(n_2-1)}$



We are given $n_1=9$, $n_2=13$, $s_1{}^2 = 1.99^2$, $s_2{}^2 = 1.9^2$
$F=\frac{9*1.99^2/(9-1)}{13*1.9^2/(13-1)} = \frac{4.455}{3.91} = 1.139$
Level of significance α=0.05

Degree of freedom $v_1 = n_1 - 1 = 8$ for the numerator
$v_2 = n_2 - 1 = 12$ for the denominator
Critical Value: The table value
$F_{(8,12)}(0.025)=3.51$
$F_{(12,8)}(0.025)=4.20$
$\frac{1}{F(12,8)(0.025)} = \frac{1}{4.20} = 0.238$
Decision: Since the calculated value F=1.139 lies between 0.238 and 3.51, we accept the null hypothesis.

(c)Find the index, rank, signature and class of the Quadratic Form $x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ by reducing it to canonical form using congruent transformation method.
Solution: **(8M)**

The matrix form is
$$A=\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix}$$
We write A=IAI
$$\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
By $R_2 - R_1, R_3 - R_1, C_2 - C_1, C_3 - C_1$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 2 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
By $R_3 - 2R_2, C_3 - 2C_2$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

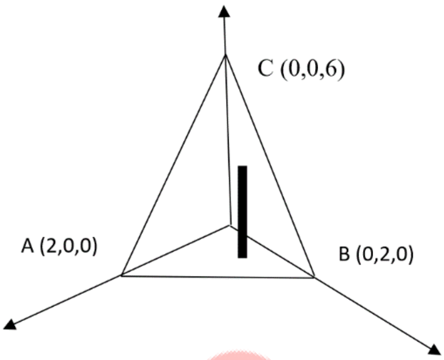
$x_1 = y_1 - y_2 + y_3$
$x_2 = y_2 - 2y_3$
$x_3 = y_3$
The rank = 3 , index = 2
Signature = difference between positive squares and negative squares = 2 – 1 = 1
Since some diagonal elements are positive, some are negative, the value class is indefinite.

Q4 (a) Evaluate $\iint_S \bar{F}.d\bar{S}$ where $\bar{F} = (2xy+z)i + y^2 j - (x+3y)k$ and S is the closed surface bounded by x=0. Y=0, z=0, 2x+2y+z=6.

Solution: **(6M)**

By divergence formula,

$$\iint_S \bar{F}.d\bar{S} = \iiint_V \nabla.\bar{F}.div$$



Now, $\bar{F} = (2xy+z)i + y^2 j - (x+3y)k$

$\nabla.\bar{F} = \frac{\delta(2xy+z)}{\delta x} + \frac{\delta(y^2)}{\delta y} - \frac{\delta(x+3y)}{\delta z}$

$= 4y$

$$\iiint_V \nabla.\bar{F}.div = \int_{x=0}^{3}\int_{y=0}^{3-x}\int_{z=0}^{6-2x-2y} 4y.\ dxdydz$$

$$= \int_{x=0}^{3}\int_{y=0}^{3-x}\int_{z=0}^{6-2x-2y} \frac{4y^2}{2}.\ dxdydz$$

$$= \int_{x=0}^{3}\int_{y=0}^{3-x}(4yz|z = 0\ to\ 6-2x-2y)\ .dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{3-x} 4y(6-2x-2y) - 4y*0\ .dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{3-x} 24y - 8xy - 8y^2\,.dx\,dy$$

$$= \int_{x=0}^{3}\left(\frac{24y^2}{2} - \frac{8xy^2}{2} - \frac{y^3}{3}\Big|x = 0\ to\ 3-x\right).dx$$

$$= \int_{x=0}^{3}\left(12y^2 - 4xy^2 - \frac{y^3}{3}\Big|x = 0\ to\ 3-x\right).dx$$

$$= \int_{x=0}^{3} 12(3-x)^2 - 4x(3-x)^2 - \frac{(3-x)^3}{3}.dx$$

$$= \int_{x=0}^{3} 12(3-x)^2 - 4(3x - 6x^2 + x^3) - \frac{(3-x)^3}{3}.dx$$

$$= \left(12\frac{(3-x)^3}{(-1)*3} - 4*\left(\frac{3x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right) - \frac{(3-x)^4}{(-1)*3*4}\Big|x = 0\ to\ 3\right)$$

$= 118.8$

(b) Verify Cayley Hamilton theorem to find $2A^4 - 5A^3 - 7A + 6$ where A= $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

Solution: **(6M)**

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(2-\lambda) = 0$

$\lambda^2 - 3\lambda - 2 = 0$

By Cayley Hamilton theorem, this equation is satisfied by A

$A^2 - 3A - 2 = 0$

Now, dividing $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$ we get

$2\lambda^4 - 5\lambda^3 - 7\lambda + 6 = (\lambda^2 - 3\lambda - 2)(2\lambda^2 + \lambda + 7) + 16\lambda + 20$

In terms of matrix A, this means

$2A^4 - 5A^3 - 7A + 6 = (A^2 - 3A - 2)(2A^2 + A + 7) + 16A + 20$

But as seen above, $A^2 - 3A - 2I = 0$

$2A^4 - 5A^3 - 7A + 6I = 16A + 20$

$2A^4 - 5A^3 - 7A + 6I = 16\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix} + 20\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}36 & 32\\ 32 & 52\end{bmatrix}$

(c) A sample of 400 students of under-graduate and 400 students of postgraduate classes was taken to know their opinion about autonomous colleges. 290 of the under-graduate and 310 of the post-graduate students favoured the autonomous status. Use chi-square test and test that the opinion regarding autonomous status of colleges is independent of the level of classes of students.

Solution: **(8M)**

	Students favoured the autonomous status	Students who have a different opinion	Total
Undergraduate	290	110	400
Post graduate	310	90	400
Total	600	200	800

(i)Null Hypothesis H_o: Opinion is independent of the level of classes (There is no association between the classes and the opinion)

Alternative Hypothesis H_a: There is association

(ii)On the basis of this hypothesis, the number in the first cell $= \frac{A\ X\ B}{N}$

Where, A=number of under-graduate students

B=number who favoured

N= Total number of students

Expected frequency $= \frac{400\ X\ 600}{800} = 300$

This is the frequency in the first cell.

The frequencies in the remaining cells are 400-300=100, 600-300=300, 400-300=100.

Calculation of $\boldsymbol{x^2}$

O	**E**	$\lvert\boldsymbol{O-E}\rvert - 0.5$	$\frac{(\lvert\boldsymbol{O-E}\rvert - 0.5)^2}{\boldsymbol{E}}$
290	300	9.5	$\frac{9.5^2}{300} = 0.301$
310	300	9.5	$\frac{9.5^2}{300} = 0.301$
110	100	9.5	$\frac{9.5^2}{100} = 0.903$
90	100	9.5	$\frac{9.5^2}{100} = 0.903$
		Total	$\boldsymbol{x^2 = 2.408}$

(iii) Level of significance : α=0.05

Degree of Freedom : $(r-1)(c-1) = (2-1)(2-1) = 1$

Critical valve : For 1 degree of freedom at 5% level of significance the table value of $x^2 = 3.81$

Decision : Since the calculated value of $x^2 = 2.408$ is less than the table value of $x^2 = 3.81$, the null hypothesis is accepted.

There is no association between the opinion and the level of classes.

Q5 (a) Prove that $\nabla x\left[\frac{\bar{a}\, X\, \bar{r}}{r^3}\right] = \frac{\bar{a}}{r^3} - \frac{3(\bar{a}\,.\,\bar{r})\,\bar{r}}{r^3}$ **(6M)**

Solution:

We have $\frac{\bar{a}\, X\, \bar{r}}{r^3} = \frac{(a_1 i+a_2 j+a_3 k)(xi+yj+zk)}{r^3} = \frac{a_1 x+a_2 y+a_3 z}{r^3}$

Let ϕ = $\frac{\bar{a}\, X\, \bar{r}}{r^3} = \frac{a_1 x+a_2 y+a_3 z}{r^3}$

$$\frac{\delta\phi}{\delta x} = \frac{a_1 r^3 - (a_1 x + a_2 y + a_3 z).3.r^2(\delta r/\delta x)}{r^6}$$

But $r^2 = x^2 + y^2 + z^2$, $2r.\frac{\delta r}{\delta x} = 2x$, $\frac{\delta r}{\delta x} = \frac{x}{r}$

$$\frac{\delta\phi}{\delta x} = \frac{a_1 r^3 - (a_1 x + a_2 y + a_3 z).3.r^1.x}{r^6} = \frac{a_1 r^3}{r^6} - \frac{(a_1 x + a_2 y + a_3 z).3.x}{r^{6-1}}$$

$$\frac{\delta\phi}{\delta x} = \frac{a_1}{r^3} - \frac{3.(a_1 x + a_2 y + a_3 z).x}{r^5}$$

$$\frac{\delta\phi}{\delta y} = \frac{a_2}{r^3} - \frac{3.(a_1 x + a_2 y + a_3 z).y}{r^5}$$

$$\frac{\delta\phi}{\delta z} = \frac{a_3}{r^3} - \frac{3.(a_1 x + a_2 y + a_3 z).z}{r^5}$$

∇ϕ = $\frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

$= \frac{1}{r^3}(a_1 i + a_2 j + a_3 k) - \frac{n}{r^5}[(a_1 x + a_2 y + a_3 z)(xi + yj + zk)]$

$\bar{a}\,.\,\bar{r} = (a_1 i + a_2 j + a_3 k)(xi + yj + zk) = (a_1 x + a_2 y + a_3 z)$

∇ϕ= $\frac{\bar{a}}{r^3} - \frac{3(\bar{a}\,.\,\bar{r})\,\bar{r}}{r^3}$

(b) Show that the matrix A=$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is diagonalizable and hence find the transforming matrix and diagonal matrix.

Solution: **(6M)**

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(3-\lambda)[(3-\lambda)(3-\lambda)-1]+1(\lambda-3+1)-1(1-3+\lambda)=0$

$\lambda^3-9\lambda^2-24\lambda+20=0$

$(\lambda-2)(\lambda^2-7\lambda+10)=0$

λ= 2, λ=2, λ= 5

for λ= 2,

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2+(-1)R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_3-R_1

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1-x_2+x_3=0$$

The rank of coefficient matrix is 1. The number of unknowns is 3. Hence, there are 3-1 = 2 linearly independent solution. Putting $x_2=t$ and $x_3=s$ then $x_1=t-s$.

$$X_1=\begin{bmatrix} t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding to the eigenvalue 2, we get the following two linearly independent eigenvectors.

$X_1=[1 \quad 1 \quad 0]'$ and $X_2=[-1 \quad 0 \quad 1]'$

for λ= 5,

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/(-2)$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - (-1)R_1$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{-3}{2} & \frac{-3}{2} \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_1$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{-3}{2} & \frac{-3}{2} \\ 1 & \frac{-3}{2} & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2/(\frac{-3}{2})$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 1 \\ 1 & \frac{-3}{2} & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (\frac{-3}{2})R_2$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 - (\frac{1}{2})R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 + x_3 = 0$$

So, $x_1 = x_3;\ x_2 = -x_3$ and $x_3 = x_3$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, A is diagonalised to $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and the diagonalizing matrix is $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(c) Ten school boys were given a test in statistics and their scores were recorded. They were given a month special coaching and a second test was given to them in the same subject at the end of the coaching period .Test at 5% level of significance, if the marks given below give evidence to the fact that the students are benefited by coaching

Mark in test I	**70**	**68**	**56**	**75**	**80**	**90**	**68**	**75**	**56**	**58**
Mark in test II	**68**	**70**	**52**	**73**	**75**	**78**	**80**	**92**	**54**	**55**

Solution: **(8M)**

We first calculate the differences between marks in test I and marks in test II = X and from these we calculate $\bar{X}$ and s^2

X	-2	2	-4	-2	-5	-12	12	17	-2	-3
$d_i = x_i - 2$	-4	0	-6	-4	-7	-14	10	15	-4	-5
${d_i}^2 = (x_i - 2)^2$	16	0	36	16	49	196	100	225	16	25

$\bar{X} = a + \frac{\sum d_i}{n} = 2 + \frac{-19}{10} = 0.1$

$\sum(X_i - \bar{X})^2 = \sum {d_i}^2 - \frac{\sum(d_i)^2}{n} = 679 - \frac{0.01}{10} = 678.999$

$\frac{\sum(X_i - \bar{X})^2}{n}$=67.90

The null hypothesis Ho µ=0

Alternative hypothesis Ha µ =/ 0

Calculation of test statistic

Since the sample size is small, we use students t-distribution

t= $\frac{\bar{X}-\mu}{s/\sqrt{n-1}} = \frac{0.1-0}{\sqrt{67.90}/\sqrt{9}} = 0.036$

Level of significance : α=0.05

Critical valve :The value of t_α at 5% level of significance for v=10-1=9 , degree of freedom=2.262

Decision : Since the calculated value of $|t| = 0.036$ is less than the critical value of t_α = 2.262, the hypothesis is accepted.

The students are not Benefitted by coaching.

Q6 (a) In the sample of 1000 cases, the mean of certain case is 14 and standard deviation is 2.5 Assuming the distribution to be normal. Find 1. How many students score between 12 and 15 2.how many score above 18. **(6M)**

Solution:

1. $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{12-14}{2.5} = -0.8$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{15-14}{2.5} = 0.4$

Area lying between -0.8 to 0.4 = Area between 0 to 0.8 + Area between 0 to 0.4

= 0.2881+0.1554

=0.4435

Required no. of students = 1000*0.4435 = 444 (approx.)

2. $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{18-14}{2.5} = 1.6$

Area right to 1.6 = 0.5 – (Area between 0 to 1.6)

=0.5 -0.4452

=0.0548

Required no. of students = 1000*0.0548 = 55 (approx.)

(b)Evaluate by Stoke's theorem $\int_c (xy\,dx + xy^2.\,dy)$ where C is the square in the xy=plane with (1,0),(0,1),(-1,0),(0,-1). **(6M)**

Solution:

By Stoke's theorem $\int_c \bar{F}.\, d\bar{r} = \iint_c \bar{N}.\nabla\, X\, \bar{F}.\, ds$

In the xy-plane $\bar{r} = xi + yj + 0k$

$\overline{dr} = dxi + dyj$

Hence, from $\bar{F}.\, d\bar{r} = (xy\,dx + xy^2.\,dy)$, we get

$\bar{F} = xy\,i + xy^2.j + 0k$

$$\nabla\, X\, \bar{F} = \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ xy & xy^2 & 0 \end{vmatrix} = (0-0)i + (0-0)j + (y^2 - x)k$$

Further, $\bar{N} = k$ and ds=dx.dy

$\iint_c \bar{N}.\nabla\, X\, \bar{F}.\, ds = \iint_S k.(y^2 - x).k\, dx.dy$

$= \iint_S (y^2 - x).dx.dy$ where S is a square ABCD

$= 4 \iint_{\Delta 0AB} (y^2 - x).dx.dy$

The equation of the line AB is $\frac{y-1}{1-0} = \frac{x-0}{0-1}$

$$\iint_{\Delta 0AB}(y^2 - x).dx.dy = \int_{x=0}^{1} \int_{y=0}^{y=1-x}(y^2 - x).dx.dy$$

$$= \int_{x=0}^{1}\left[\left(\frac{y^3}{3} - xy\middle| y = 0\ to\ 1 - x\right)\right].dx$$

$$= \int_0^1 \frac{(1-x)^3}{3} - x(1-x)\,.\,dx$$

$$= \frac{1}{3}\left(x - 3x^2 + 2x^3 - \frac{x^4}{4}\middle| x = 0\ to\ 1\right)$$

$$= \frac{1}{3} * \left(1 - 3 + 2 - \frac{1}{4}\right) = -1/12$$

$$\int_c \overline{F}.\ d\overline{r} = \iint_c \overline{N}.\nabla\ X\ \overline{F}.\ ds = 4 * \left(-\frac{1}{12}\right) = -1/3$$

(c) Solve the L.P.P. from its primal as well as from its dual

Minimise $z=0.7x_1+0.5x_2$

Subjected to $x_1 \geq 4$, $x_2 \geq 6$

$x_1+2x_2 \geq 20$

$2x_1+ x_2 \geq 18$

$x_1, x_2 \geq 0$ **(8M)**

Solution:

Maximize $z' = -z = -0.7x_1-0.5x_2- 0s_1 -0s_2 -0s_3 -0s_4 -MA_1 -MA_2 -MA_3 -MA_4$ ………..(1)

Subjected to $x_1+0x_2- s_1 +0s_2 +0s_3 +0s_4 +A_1 +0A_2 +0A_3 +0A_4=4$…………..(2)

$0x_1+x_2+s_1 - s_2 +0s_3 +0s_4 +0A_1 +A_2 +0A_3 +0A_4=6$…………..(3)

$x_1+2x_2 +0s_1 +0s_2 -s_3 +0s_4 +0A_1 +0A_2 +A_3 +0A_4=20$………..(4)

$2x_1+x_2- 0s_1 +0s_2 +0s_3 -s_4 +0A_1 +0A_2 +0A_3 +A_4=18$…………(5)

Multiply (1),(2),(3) and (4) by M and to (1)

$z'= (-0.7+4M)x_1+(-0.5+M)x_2- Ms_1 -Ms_2 -Ms_3 -Ms_4 -0A_1 -0A_2 -0A_3 -0A_4 -48M$

$z'+ (0.7-4M)x_1+(0.5-M)x_2+ Ms_1 +Ms_2 +Ms_3 +Ms_4 +0A_1 +0A_2 +0A_3 +0A_4 =-48M$

Iteration No.	Basic Variable	Coef	ficie	nts			of					R.H.S. Solution	Ratio
		x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4		
0	z	0.7-4M	0.5-4M	M	M	M	M	0	0	0	0	-48M	
A_2 leaves	A_1	1	0	-1	0	0	0	1	0	0	0	4	-
x_2 enters	A_2	0	1*	0	-1	0	0	0	1	0	0	6	6
	A_3	1	2	0	0	-1	0	0	0	1	0	20	10
	A_4	2	1	0	0	0	-1	0	0	0	1	18	18

1	z	0.7-4M	0	M	0.5-3M	M	M	0		0	0	-3-24M	
A_2 leaves	A_1	1*	0	-1	0	0	0	1		0	0	4	4
$x_1$enters	x_2	0	1	0	-1	0	0	0		0	0	6	-
	A_3	1	0	0	2	-1	0	0		1	0	8	8
	A_4	2	0	0	1	0	-1	0		0	0	12	6
2	z	0	0	0.7-3M	0.5-3M	M	M			0	0	-5.8-8M	
A_3 leaves	x_1	1	0	-1	0	0	0			0	0	4	-
$s_2$enters	x_2	0	1	0	-1	0	0			0	0	6	-
	A_3	0	0	1	2*	-1	0			1	0	4	2
	A_4	0	0	2	1	0	-1			0	0	4	4
3	z	0	0	0.45-3/2M	0	0.5/2-M/2	M				0	-6.8-2M	
A_4 leaves	x_1	1	0	-1	0	0	0				0	4	-
$s_1$enters	x_2	0	1	½	0	-1/2	0				0	8	16
	s_2	0	0	½	1	-1/2	0				0	2	4
	A_4	0	0	3/2*	0	1/2	-1				0	2	4/3
4	z	0	0	0	0	0.1	0.3					-7.4	
	x_1	1	0	0	0	1/3	-2/3					16/3	
	x_2	0	1	0	0	-2/3	1/3					22/3	
	s_2	0	0	0	1	-2/3	1/3					4/3	
	s_1	0	0	1	0	1/3	-2/3					4/3	